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Strategic Claim Games Corresponding to an NTU-Game

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This paper introduces a way to construct a game $\Gamma(V)$ in strategic form to a standard NTU-game V . In the strategic claim game $\Gamma(V)$ the players are allowed to make claims on the coalition they want to participate in, and on the payoff they want to attain. Conversely, to construct an NTU-game to each game in strategic form the paper follows R. J. Aumann (1961, in *Contributions to the Theory of Games* (A. W. Tucker and R. D. Luce, Eds.), Vol. IV, pp. 287–324. Princeton, NJ: Princeton Univ. Press), but instead of correlation only coordination of the players' actions is allowed. Applying this procedure to the claim game $\Gamma(V)$, an NTU-game V is associated with an NTU-game \bar{V} . It is found that \bar{V} is the superadditive hull of V . Further, it is shown that if V is superadditive, strong core elements of V exactly correspond to the payoff vectors of strong Nash equilibria for $\Gamma(V)$. Finally, some applications of the Aumann procedure are discussed. *Journal of Economic Literature* Classification Numbers: C71, C72. © 1992 Academic Press, Inc.

1. INTRODUCTION

In 1944 von Neumann and Morgenstern indicated a way to construct a superadditive game in *coalitional form* to each game in *strategic form*. It was proved that in this way all superadditive games in coalitional form could be obtained. More generally, Aumann (1961, 1967) constructed a superadditive *NTU-game* (NTU = nontransferable utility) to each game in strategic form. In both of these procedures the players are allowed to act highly cooperative in the sense that they can *correlate* their actions. In particular, this implies that Aumann's procedure always leads to NTU-games in which all sets of attainable payoff vectors for the various coalitions are convex. In this paper we adapt Aumann's procedure by only allowing the players to *coordinate* their actions. Again, the resulting NTU-game is superadditive. The starting point for our research was the following question. If one does not require convexity in the definition of



an NTU-game, can all superadditive NTU-games be obtained from games in strategic form by using the “coordinated” Aumann procedure?

The “almost” affirmative answer to this question is found by defining a strategic *claim game* $\Gamma(V)$ to each *standard* NTU-game V . The construction of a claim game bears some resemblance to the way Nash (1950, 1953) constructed a game in strategic form to each two-person bargaining game, and also to the way von Neumann and Morgenstern (1944) showed that each superadditive game in coalitional form arises from a game in strategic form. According to Nash both players can make claims on their own payoffs. If this pair of claim fits (i.e., is a feasible pair), the players get their claims. If not, the disagreement point is reached. Von Neumann and Morgenstern allowed the players to make claims on the coalition they want to participate. A coalition S forms if all the players in S claim S . If S is formed, each of the players in S obtains an equal share of the worth of S . If a player's claim is not consistent in the above sense, he obtains the payoff he can make on his own. In a claim game $\Gamma(V)$ of Section 2 the players are allowed to make claims on both a coalition and a payoff.

Applying the coordinated Aumann procedure to the claim game $\Gamma(V)$ the *claim associate* \bar{V} of V is obtained. In Section 2 it is shown that the claim associate \bar{V} is the *superadditive hull* of V , i.e., the smallest superadditive game containing V . This especially implies that the class of standard superadditive NTU-games can be seen as arising from games in strategic form.

Subsequently, Section 3 concentrates on the implementation of cooperative solution concepts (cf. Ichiishi, 1987). It is shown that if V is superadditive, imputations and strong core elements of V exactly correspond to the payoff vectors that are obtained by payoff undominated equilibria and strong equilibria of $\Gamma(V)$, respectively.

Finally, in Section 4, it is seen that the NTU-game \tilde{V} , which is obtained by applying the correlated Aumann procedure to the claim game $\Gamma(V)$, is the convex hull of the claim associate \bar{V} . Further, we discuss the possibility of defining *finite* claim games for a special class of NTU-games.

Notation. Let $x, y \in \mathbf{R}^t$. Then $x \geq y$ ($x > y$) if $x_i \geq y_i$ ($x_i > y_i$) for all $i \in \{1, \dots, t\}$. For a finite set A , let $|A|$ denote the number of elements of A and let $\langle A \rangle$ denote the set of all possible partitions of A , i.e.,

$$\langle A \rangle := \left\{ \{A_1, \dots, A_r\} \mid r \in \mathbf{N}, \bigcup_{k=1}^r A_k = A, A_k \cap A_l = \emptyset \right. \\ \left. \text{for all } k, l \in \{1, \dots, r\} \text{ with } k \neq l \right\}.$$

Finally, for a convex set $C \subset \mathbf{R}^t$, $\text{Ext}(C)$ denotes the set of extreme points of C . Here, $c \in C$ is called extreme if for all $c_1, c_2 \in C$ with $c = \frac{1}{2}(c_1 + c_2)$ we have that $c_1 = c_2$.

2. NTU-GAMES, CLAIM GAMES, AND CLAIM ASSOCIATES

An *NTU-game* is an ordered pair (N, V) , where $N := \{1, \dots, n\}$ is the set of players and, with $2^N := \{S \mid S \subset N\}$ representing the set of possible coalitions, V is a mapping which assigns to each coalition $S \in 2^N \setminus \{\emptyset\}$ a subset $V(S)$ of \mathbf{R}^S such that $V(S)$ is *comprehensive*, i.e., if $a \in V(S)$ and $b \in \mathbf{R}^S$ is such that $b \leq a$, then $b \in V(S)$.

This can be interpreted as follows. If a coalition S forms, each of the payoff vectors $a \in V(S)$ is attainable by S , giving a payoff (utility) a_i to player $i \in S$. We often identify (N, V) with V . The set of all n -person NTU-games is denoted by NTU^n . A game $V \in \text{NTU}^n$ is called *superadditive* if

$$V(S) \times V(T) \subset V(S \cup T) \quad (1)$$

for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \cap T = \emptyset$

Note that each n -person game (N, v) in *coalitional form* with $v: 2^N \rightarrow \mathbf{R}$ and $v(\emptyset) = 0$ gives rise to an NTU-game (N, V) , where

$$V(S) := \left\{ a \in \mathbf{R}^S \mid \sum_{i \in S} a_i \leq v(S) \right\} \quad (S \in 2^N \setminus \{\emptyset\}).$$

Another example is provided by two-person *bargaining games* (C, d) , where C is a compact subset of \mathbf{R}^2 representing the feasible utility pairs and $d \in \mathbf{R}^2$ is the disagreement point. Such a bargaining game can be viewed as an NTU-game $(\{1, 2\}, V)$ by defining

$$V(\{i\}) = \{a \in \mathbf{R} \mid a \leq d_i\} \text{ for } i \in \{1, 2\} \quad \text{and} \quad V(\{1, 2\}) = \text{Compr}(C),$$

where $\text{Compr}(A) := \{b \in \mathbf{R}^t \mid \text{there is an } a \in A \text{ such that } b \leq a\}$ denotes the *comprehensive hull* of a subset A of \mathbf{R}^t .

For an n -person game $\Gamma = (X_1, \dots, X_n, K_1, \dots, K_n)$ in strategic form, X_i and $K_i: X_1 \times \dots \times X_n \rightarrow \mathbf{R}$ represent the strategy space and the payoff function of player $i \in \{1, \dots, n\}$, respectively. The set of all n -person games in strategic form is denoted by SG^n . Furthermore, for $S \in 2^N \setminus \{\emptyset\}$ and $x = (x_1, \dots, x_n) \in \prod_{i \in N} X_i$,

$$X_S := \prod_{i \in S} X_i, x_S := (x_i)_{i \in S} \in X_S, K_S(x) := (K_i(x))_{i \in S} \in \mathbf{R}^S$$

and x is identified with $(x_S, x_{N \setminus S})$.

Following Aumann (1961, 1967), but allowing the players only to coordinate their actions, we construct NTU-games A_Γ and B_Γ to each game Γ in strategic form in the following way.

DEFINITION. Let $\Gamma = (X_1, \dots, X_n, K_1, \dots, K_n) \in SG^n$. Then $A_\Gamma \in NTU^n$ and $B_\Gamma \in NTU^n$ are defined by

$$A_\Gamma(S) := \{a \in \mathbf{R}^S \mid \exists_{\hat{x}_S \in X_S} \forall_{x_{N \setminus S} \in X_{N \setminus S}} : K_S(\hat{x}_S, x_{N \setminus S}) \geq a\} \quad (2)$$

and

$$B_\Gamma(S) := \{a \in \mathbf{R}^S \mid \forall_{x_{N \setminus S} \in X_{N \setminus S}} \exists_{\hat{x}_S \in X_S} : K_S(\hat{x}_S, x_{N \setminus S}) \geq a\} \quad (3)$$

for all $S \in 2^N \setminus \{\emptyset\}$.

Note that $A_\Gamma(S)$ corresponds to a pessimistic view on the cooperative behavior of the coalition S , while $B_\Gamma(S)$ corresponds to a more optimistic point of view.

EXAMPLE 1. Consider the two-person game Γ in strategic form with strategy spaces $\{T, B\}$ for player 1 and $\{L, R\}$ for player 2, and where the payoffs are determined by

$$\begin{array}{cc} & \text{player 2} \\ & \begin{array}{cc} L & R \end{array} \\ \text{player 1} & \begin{array}{cc} T & \begin{pmatrix} (1, -1) & (-1, 1) \end{pmatrix} \\ B & \begin{pmatrix} (-1, 1) & (1, -1) \end{pmatrix} \end{array} \end{array}$$

Then

$$\begin{aligned} A_\Gamma(\{1\}) &= A_\Gamma(\{2\}) = \{a \in \mathbf{R} \mid a \leq -1\}, \\ B_\Gamma(\{1\}) &= B_\Gamma(\{2\}) = \{a \in \mathbf{R} \mid a \leq 1\} \end{aligned}$$

and

$$A_\Gamma(\{1, 2\}) = B_\Gamma(\{1, 2\}) = \text{Compr}(\{(-1, 1), (1, -1)\}).$$

Note that B_Γ is not superadditive.

It is straightforward to show

LEMMA 2.1. A_Γ is superadditive for all $\Gamma \in SG^n$.

Conversely, for a large class of NTU-games, we can define a corresponding game in strategic form in the following way.

DEFINITION. Let $V \in NTU^n$ be called *standard* if $V(\{i\})$ is nonempty, upper-bounded, and closed for all $i \in N$.

For a standard NTU-game V , the corresponding *claim game* $\Gamma(V) \in SG^n$ is given by $\Gamma(V) = (X_1, \dots, X_n, K_1, \dots, K_n)$, where for each $i \in N$

$$X_i := \{S \in 2^N \mid i \in S\} \times \mathbf{R} \quad (4)$$

and for $x = (S_j, t_j)_{j \in N} \in \prod_{j \in N} X_j$

$$K_i(x) = \begin{cases} t_i & \text{if } S_j = S_i \text{ for all } j \in S_i \text{ and } (t_j)_{j \in S_i} \in V(S_i) \\ \min\{t_i, v(i)\} & \text{otherwise} \end{cases} \quad (5)$$

with $v(i) := \max(V(\{i\}))$.

In a claim game $\Gamma(V)$ the strategy (S_i, t_i) of player $i \in N$ can be interpreted as follows. Player i wants the coalition $S_i \ni i$ to form and he claims a payoff equal to t_i . His demands are met if the wishes with respect to the coalition formation are consistent for all players in S_i (i.e., $S_j = S_i$ for all $j \in S_i$) and the payoff (demand) vector for S_i is attainable in V (i.e., $(t_j)_{j \in S_i} \in V(S_i)$). If one of these two conditions is violated, player i in principle gets his own “individual rational” payoff $v(i)$. However, if his claim t_i is less than $v(i)$, he obtains a payoff equal to t_i .

DEFINITION. Let $V \in \text{NTU}^n$ be standard. Let $x = (S_i, t_i)_{i \in N}$ be a strategy vector in the claim game $\Gamma(V)$. Then the *fitting* $F_x \subset 2^N \setminus \{\emptyset\}$ is the partition of N , defined as follows. For $S \subset N$ with $|S| \geq 2$ we have that $S \in F_x$ if

$$S_i = S \text{ for all } i \in S \quad \text{and} \quad (t_i)_{i \in S} \in V(S).$$

Further, with $i \in N$, we have that $\{i\} \in F_x$ if and only if

$$i \notin \bigcup \{S \in F_x \mid |S| \geq 2\}.$$

Consequently,

$$K_T(x) \in V(T) \quad \text{for all } T \in F_x. \quad (6)$$

Claim games are *tight* (cf. Ichiishi, 1986, p. 281) in the sense that the associated NTU-games of (2) and (3) are identical. This is seen in

LEMMA 2.2. *Let $V \in \text{NTU}^n$ be standard. Then $A_{\Gamma(V)} = B_{\Gamma(V)}$.*

Proof. Let $S \in 2^N \setminus \{\emptyset\}$ and $\Gamma(V) = (X_1, \dots, X_n, K_1, \dots, K_n)$. Since it is clear that $A_{\Gamma(V)}(S) \subset B_{\Gamma(V)}(S)$, it suffices to show the converse.

Let $b \in B_{\Gamma(V)}(S)$. Let $\hat{x}_S = (S_i, t_i)_{i \in S} \in X_S$ be such that

$$K_S(\hat{x}_S, (\{j\}, v(j))_{j \in N \setminus S}) \geq b.$$

Then, without loss of generality, we may assume that $S_i \subset S$ for all $i \in S$. Consequently, for all $x_{N \setminus S} \in X_{N \setminus S}$,

$$T \cap S \neq \emptyset, T \in F_{(\hat{x}_S, x_{N \setminus S})} \Rightarrow T \in F_{(\hat{x}_S, (\{j\}, v(j))_{j \in N \setminus S})}$$

and so

$$K_S(\hat{x}_S, x_{N \setminus S}) = K_S(\hat{x}_S, (\{j\}, v(j))_{j \in N \setminus S}) \geq b.$$

Hence, $b \in A_{\Gamma(V)}(S)$. ■

DEFINITION. Let $V \in \text{NTU}^n$ be standard. Then the *claim associate* $\bar{V} \in \text{NTU}^n$ is defined by $\bar{V} := A_{\Gamma(V)} (= B_{\Gamma(V)})$.

It is easily verified that a claim associate \bar{V} is standard because $\bar{V}(\{i\}) = V(\{i\})$ for all $i \in N$. Theorem 2.1 below shows that the claim associate \bar{V} is the superadditive hull of V . In particular this implies that each standard superadditive NTU-game can be seen as arising from a (claim) game in strategic form.

THEOREM 2.1. Let $V \in \text{NTU}^n$ be standard. Then the claim associate \bar{V} is the smallest superadditive game containing V .

Proof. Let $\Gamma(V) = (X_1, \dots, X_n, K_1, \dots, K_n)$ and $S \in 2^N \setminus \{\emptyset\}$. Then, clearly, $V(S) \subset \bar{V}(S)$ because, with $a \in V(S)$ and $\hat{x}_S := (S, a_i)_{i \in S} \in X_S$, it is found that $K_S(\hat{x}_S, x_{N \setminus S}) = a$ for all $x_{N \setminus S} \in X_{N \setminus S}$, which implies that $a \in \bar{V}(S)$.

Let $W \in \text{NTU}^n$ be superadditive and assume that $V \subset W$. For proving the theorem, it suffices to show that $\bar{V} \subset W$. First we make the following observations. Let $V^1 \in \text{NTU}^n$ be standard and let $\Gamma(V^1) = (X_1, \dots, X_n, L_1, \dots, L_n)$.

(a) If $V \subset V^1$, then $\bar{V} \subset \bar{V}^1$. This immediately follows from the fact that $K_N(x) \leq L_N(x)$ for all $x \in X_N$ (cf. (5)).

(b) If V^1 is superadditive then $\bar{V}^1 = V^1$. For this, it suffices to show that $\bar{V}^1(S) \subset V^1(S)$. Let $a \in \bar{V}^1(S)$ and let $\bar{x}_S \in X_S$ be such that

$$L_S(\bar{x}_S, (\{j\}, v(j))_{j \in N \setminus S}) \geq a.$$

It is clear that the fitting $F_{(\bar{x}_S, (\{j\}, v(j))_{j \in N \setminus S})}$ induces a partition $\{S(1), \dots, S(r)\}$ of S such that, according to (6), $a_{S(k)} = (a_i)_{i \in S(k)} \in V^1(S(k))$ for all $k \in \{1, \dots, r\}$. Hence,

$$a = (a_{S(k)})_{k \in \{1, \dots, r\}} \in \prod_{k=1}^r V^1(S(k)) \subset V^1(S)$$

by superadditivity of V^1 . Consequently, $a \in V^1(S)$.

Choosing $V^1 \in \text{NTU}^n$ such that $V^1(\{i\}) = V(\{i\})$ for all $i \in N$ and $V^1(S) = W(S)$ for all $S \in 2^N$ with $|S| \geq 2$, (a) and (b) imply that

$$\bar{V} \subset \bar{V}^1 = V^1 \subset W.$$

This finishes the proof. ■

Theorem 2.1 straightforwardly implies the following “inside” description for a claim associate.

COROLLARY 2.2. *Let $V \in \text{NTU}^n$ and $S \in 2^N \setminus \{\emptyset\}$. Then*

$$\bar{V}(S) = \bigcup_{P \in \langle S \rangle} \prod_{T \in P} V(T).$$

In a strategic claim game $\Gamma(V)$ it may seem unnatural for player i to ever claim a payoff that is less than his individual rational level $v(i)$. By shrinking the strategy spaces in order to rule these claims out in advance, one obtains a *restricted claim game* $\Gamma^r(V)$ and a corresponding *restricted claim associate* \bar{V}^r . It is not difficult to verify the following relation between the claim associates \bar{V} and \bar{V}^r . For all (nonempty) coalitions S , it holds that

$$\bar{V}^r(S) = \text{Compr}(\{a \in \bar{V}(S) \mid a \geq (v(i))_{i \in S}\}). \quad (7)$$

3. IMPLEMENTATION OF COOPERATIVE SOLUTION CONCEPTS

First we recall the definitions of some well-known solution concepts for NTU-games.

DEFINITION. Let $V \in \text{NTU}^n$ be standard. The *imputation set* $I(V)$ is defined by

$$I(V) := \{a \in \mathbf{R}^n \mid a \in U(V(N)), a \geq (v(i))_{i \in N}\},$$

where, for $A \subset \mathbf{R}^n$, $U(A)$ denotes the set of *undominated elements* of A , i.e.,

$$U(A) := \{a \in A \mid \neg \exists b \in A : b \geq a \text{ and } b \neq a\}.$$

The *strong core* $SC(V)$ consists of those payoff vectors in $V(N)$ which are “stable” with respect to domination for each subcoalition S . More specifically,

$$SC(V) := \{a \in V(N) \mid \neg \exists_{S \in 2^{N \setminus \{0\}}} : a_S \in V(S) \setminus U(V(S))\}.$$

The *core* $C(V)$ consists of those payoff vectors in $V(N)$ which are “stable” with respect to *strict* domination, i.e.,

$$C(V) := \{a \in V(N) \mid \neg \exists_{S \in 2^{N \setminus \{0\}}} \exists_{b \in V(S)} : b > a_S\}.$$

From these definitions it is clear that the strong core is a subset of both the core and the imputation set. Furthermore, for a two-person (standard) NTU-game the imputation set and the strong core coincide.

Now we recall the definitions of the various types of equilibria for games in strategic form that play a role in implementing imputations and strong core elements.

DEFINITION. Let $\Gamma = (X_1, \dots, X_n, K_1, \dots, K_n) \in SG^n$. The set $E(\Gamma)$ of *Nash equilibria* for Γ is defined by

$$E(\Gamma) := \{x \in X_N \mid \neg \exists_{i \in N} \exists_{y_i \in X_i} : K_i(y_i, x_{N \setminus \{i\}}) > K_i(x)\},$$

the set $SE(\Gamma)$ of *strong Nash equilibria* by

$$SE(\Gamma) := \{x \in X_N \mid \neg \exists_{S \in 2^{N \setminus \{0\}}} \exists_{y_S \in X_S} : K_S(y_S, x_{N \setminus S}) \geq K_S(x) \text{ and } K_S(y_S, x_{N \setminus S}) \neq K_S(x)\},$$

and the set $PUE(\Gamma)$ of *payoff undominated equilibria* by

$$PUE(\Gamma) := \{x \in E(\Gamma) \mid \neg \exists_{y \in X_N} : K_N(y) \geq K_N(x) \text{ and } K_N(y) \neq K_N(x)\}.$$

In Theorem 3.1 below it is seen that each imputation for a (standard) NTU-game V comes forward as a payoff vector attained by a Nash equilibrium for the claim game $\Gamma(V)$. Moreover, if V is superadditive, then the imputation set $I(V)$ is equal to the set of payoff vectors attained by payoff undominated equilibria for $\Gamma(V)$.

THEOREM 3.1. *Let $V \in NTU^n$ be standard and $\Gamma(V) = (X_1, \dots, X_n, K_1, \dots, K_n)$. Then the following two assertions holds.*

- (i) *For all $a \in I(V)$ there is an $x \in E(\Gamma(V))$ such that $K_N(x) = a$.*
- (ii) *If V is superadditive, then $a \in I(V)$ if and only if there exists an $x \in PUE(\Gamma(V))$ such that $K_N(x) = a$.*

Proof. (i) Let $a \in I(V)$. Define $x := (N, a_j)_{j \in N} \in X_N$. It is clear that $K_N(x) = a$. Suppose $x \notin E(\Gamma(V))$. Then there is a player $i \in N$ and a strategy $y_i = (S_i, t_i) \in X_i$ such that

$$t_i = K_i(y_i, x_{N \setminus \{i\}}) > a_i \geq v(i).$$

Hence, $S_i = N$ and $b := (t_i, (a_j)_{j \in N \setminus \{i\}}) \in V(N)$. However, this contradicts the fact that $a \in U(V(N))$ since $b \geq a$ and $b \neq a$.

(ii) Let V be superadditive. First assume $a \in I(V)$. By defining $x := (N, a_i)_{i \in N}$ we have (cf. (i)) $K_N(x) = a$ and $x \in E(\Gamma(V))$. Suppose $x \notin \text{PUE}(\Gamma(V))$. Then there exists a strategy vector $y \in X_N$ such that $K_N(y) \geq a$ and $K_N(y) \neq a$. Using (6) we can find a partition $\{N(1), \dots, N(r)\}$ of N such that

$$K_N(y) \in \prod_{k=1}^r V(N(k)) \subset V(N),$$

by superadditivity of V . However, this should imply that $a \notin U(V(N))$. Conversely, let $x \in \text{PUE}(\Gamma(V))$ and define $a := K_N(x)$. Trivially, $a \geq (v(i))_{i \in N}$. Superadditivity and (6) imply that $a \in V(N)$. It remains to prove that a is undominated in $V(N)$. Suppose we can find a vector $b \in V(N)$ such that $b \geq a$, $b \neq a$. Defining $y := (N, b_i)_{i \in N} \in X_N$ we find that $K_N(y) = b \geq a = K_N(x)$. Since $b \neq a$, this contradicts the fact that x is payoff undominated. ■

Now we show that each strong core element for a (standard) NTU-game V comes forward as a payoff vector attained by a strong Nash equilibrium for the claim game $\Gamma(V)$. Moreover, if V is superadditive, then each payoff vector attained by a strong equilibrium corresponds to a strong core element.

THEOREM 3.2. *Let $V \in \text{NTU}^n$ be standard and $\Gamma(V) = (X_1, \dots, X_n, K_1, \dots, K_n)$. Then the following two assertions hold.*

- (i) *For all $a \in \text{SC}(V)$ there is an $x \in \text{SE}(\Gamma(V))$ such that $K_N(x) = a$.*
- (ii) *If V is superadditive, then $a \in \text{SC}(V)$ if and only if there exists an $x \in \text{SE}(\Gamma(V))$ such that $K_N(x) = a$.*

Proof. We only prove (i). For the proof of (ii) one can use a similar line of argument as in the corresponding part of the proof of Theorem 3.1(ii).

Let $a \in \text{SC}(V)$ and define $x := (N, a_j)_{j \in N}$. Suppose $x \notin \text{SE}(\Gamma(V))$. Then there exists a coalition $S \in 2^N \setminus \{\emptyset\}$, a player $i \in S$, and a strategy vector $y_S = (S_j, t_j)_{j \in S} \in X_S$ such that

$$t_S = K_S(y_S, x_{N \setminus S}) \geq a_S \geq (v(j))_{j \in S} \quad \text{and} \quad t_i > a_i$$

Hence, $S_i \in F_{(y_S, x_{N \setminus S})}$ and $|S_i| \geq 2$. If $S_i \subset S$, then (6) implies that

$$V(S_i) \ni K_{S_i}(y_S, x_{N \setminus S}) = t_{S_i} \geq a_{S_i}.$$

However, since $t_i > a_i$, this would imply that $a_{S_i} \notin U(V(S_i))$ and contradict the fact that $a \in SC(V)$. So we may assume that $(N \setminus S) \cap S_i \neq \emptyset$. Consequently, $S_i = N$ and

$$v(N) \ni K_N(y_S, x_{N \setminus S}) = (t_S, a_{N \setminus S}) \geq a.$$

This, however, would imply that $a \notin U(V(N))$. ■

In particular, Theorem 3.2 implies that, for two-person bargaining games, the Nash bargaining solution (which by definition is an element of the strong core) corresponds to a payoff vector attained by a (strong) Nash equilibrium of a (claim) game in a strategic form.

Now we illustrate that core elements cannot be implemented in the same way as imputations and strong core elements. More specifically, the superadditive game V of Example 2 has core elements which do not correspond to any payoff vector attained by a Nash equilibrium for the claim game $\Gamma(V)$.

EXAMPLE 2. Let the two-person NTU-game V be given by

$$V(\{1\}) = V(\{2\}) = \{a \in \mathbf{R} \mid a \leq 0\}$$

and

$$V(\{1, 2\}) = \{(a, b) \in \mathbf{R}^2 \mid a \leq 5, b \leq 5, a + b \leq 6\}.$$

Then $(5, 0) \in C(V)$, while the set of all payoff vectors attained by Nash equilibria for $\Gamma(V)$ equals

$$\{(0, 0)\} \cup \{(a, b) \in \mathbf{R}^2 \mid a + b = 6, a \geq 1, b \geq 1\}.$$

Remark. It may be noted that in all results of this section one can replace $\Gamma(V)$ by the restricted claim game $\Gamma^r(V)$ because the various types of equilibria for $\Gamma(V)$ and $\Gamma^r(V)$ coincide. Further, using appropriate definitions, it is possible to extend all results toward ε -(strong) cores, ε -(strong) Nash equilibria, etc.

4. CORRELATION AND FINITE CLAIM GAMES

In this section we follow Aumann (1961, 1967) and allow correlation of the players' actions in the construction of an NTU-game to a game in strategic form.

DEFINITION. Let $\Gamma = (X_1, \dots, X_n, K_1, \dots, K_n) \in SG^n$. For $S \subset N$, the set C_S of *correlated strategies* for S in Γ is defined by

$$C_S := \left\{ \sum_{k=1}^r \lambda_k \delta(x_S^k) \mid r \in \mathbf{N}, \sum_{k=1}^r \lambda_k = 1, \lambda_k \geq 0 \right. \\ \left. \text{and } x_S^k \in X_S \text{ for all } k \in \{1, \dots, r\} \right\},$$

where, for each $x_S \in X_S$, $\delta(x_S)$ represents the probability measure on X_S which assigns probability one to x_S . So correlated strategies for S in Γ correspond to probability measures on X_S with finite support. Allowing correlated strategies, the payoff functions K_i , $i \in N$, are extended in the following natural way. For $S \subset N$, $c_S = \sum_{k=1}^r \lambda_k \delta(x_S^k) \in C_S$, and $c_{N \setminus S} = \sum_{l=1}^s \mu_l \delta(x_{N \setminus S}^l) \in C_{N \setminus S}$, we define

$$K_S(c_S, c_{N \setminus S}) := \sum_{k=1}^r \sum_{l=1}^s \lambda_k \mu_l K_S(x_S^k, x_{N \setminus S}^l).$$

Further, the NTU-game \tilde{A}_Γ corresponding to Γ is defined by

$$\tilde{A}_\Gamma(S) := \{a \in \mathbf{R}^S \mid \exists \hat{c}_S \in C_S \forall c_{N \setminus S} \in C_{N \setminus S} : K_S(\hat{c}_S, c_{N \setminus S}) \geq a\} \quad (8)$$

for all $S \in 2^N \setminus \{\emptyset\}$.

It is not difficult to verify that $A_\Gamma \subset \tilde{A}_\Gamma$, \tilde{A}_Γ is superadditive and $\tilde{A}_\Gamma(S)$ is convex for all coalitions S . Further, if one defines \tilde{B}_Γ in the obvious way (cf. (3)), then the same arguments as in the proof of Lemma 2.2 imply that $\tilde{A}_{\Gamma(V)} = \tilde{B}_{\Gamma(V)}$ for all standard NTU-games V .

DEFINITION. Let $V \in \text{NTU}^n$ be standard. Then the *correlated claim associate* $\tilde{V} \in \text{NTU}^n$ is defined by $\tilde{V} = \tilde{A}_{\Gamma(V)} (= \tilde{B}_{\Gamma(V)})$.

THEOREM 4.1. Let $V \in \text{NTU}^n$ be standard. Then \tilde{V} is the convex hull of \bar{V} .

Proof. Let $S \in 2^N \setminus \{\emptyset\}$. Clearly, $\bar{V}(S) \subset \tilde{V}(S)$. So, by convexity of $\tilde{V}(S)$, $\text{Conv}(\bar{V}(S)) \subset \tilde{V}(S)$. To prove the converse, let $a \in \tilde{V}(S)$. Define $W \in \text{NTU}^n$ by $W := \text{Conv}(\bar{V})$. Then W is superadditive because \bar{V} is, and $W(\{i\}) = \bar{V}(\{i\}) = V(\{i\})$ for all $i \in N$. Further, since $V \subset W$, we have that $\tilde{V} \subset \tilde{W}$. Hence, $a \in \tilde{W}(S)$ and therefore, with $\Gamma(W) = (X_1, \dots, X_n, K_1, \dots, K_n)$, there exists a correlated strategy vector $c_S = \sum_{k=1}^r \lambda_k \delta(x_S^k) \in C_S$ such that

$$a \leq K_S(c_S, (\{j\}, v(j))_{j \in N \setminus S}) \\ = \sum_{k=1}^r \lambda_k K_S(x_S^k, (\{j\}, v(j))_{j \in N \setminus S}) \in \text{Conv}(W(S)) = W(S),$$

because the superadditivity of W and (6) imply that $K_S(x_S^k, (\{j\}, v(j))_{j \in N \setminus S}) \in W(S)$ for all $k \in \{1, \dots, r\}$. Consequently, $a \in W(S)$. ■

In particular, Theorem 4.1 implies that all (standard) superadditive NTU-games with convex sets of attainable payoff vectors can be obtained from games in strategic form by means of the correlated Aumann procedure.

Now we introduce a special class of NTU-games V for which it is possible to define a *finite* claim game $\Gamma^f(V)$ in such a way that by applying the (correlated) Aumann procedure to $\Gamma^f(V)$, we again obtain the correlated claim associate \bar{V} that was originally derived from the infinite claim game $\Gamma(V)$.

DEFINITION. A game $V \in \text{NTU}^n$ is called *finitely generated* if, for all $S \in 2^N \setminus \{\emptyset\}$, $V(S)$ is convex, $1 \leq |\text{Ext}(V(S))| < \infty$, and

$$V(S) = \text{Comp}(\text{Conv}(\text{Ext}(V(S)))). \quad (9)$$

Note that each finitely generated game is standard. Let $V \in \text{NTU}^n$ be finitely generated and $\Gamma(V) = (X_1, \dots, X_n, K_1, \dots, K_n)$.

The *finite claim game* $\Gamma^f(V)$ corresponding to V is given by $\Gamma^f(V) = (X_1^f, \dots, X_n^f, K_1, \dots, K_n)$, where, for each $i \in N$,

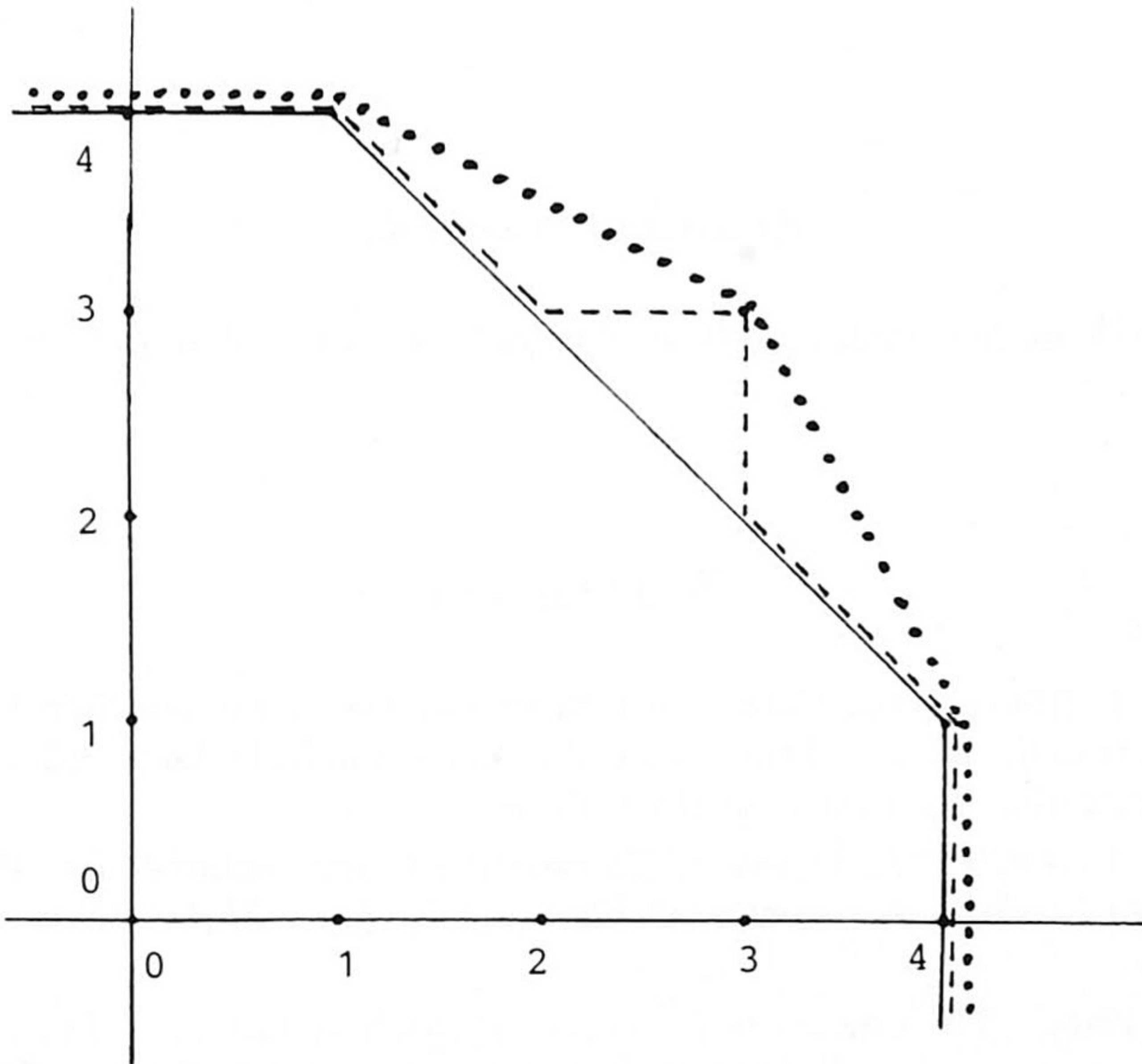


FIG. 1. (—) Boundary of $V(\{1, 2\})$; (---) boundary of $\bar{V}(\{1, 2\})$; (···) boundary of $\tilde{V}(\{1, 2\})$.

$$X_i^f := \bigcup_{S \in 2^N \setminus \{\emptyset\}: S \ni i} \{(S, t) \mid t = a_i \text{ for some } a \in \text{Ext}(V(S))\} \quad (10)$$

and the payoff function K_i is restricted to $X_N^f \subset X_N$.

Without proof we state

THEOREM 4.2. *Let $V \in \text{NTU}^n$ be finitely generated. Then $\tilde{V} = \tilde{A}_{\Gamma(V)} = \tilde{A}_{\Gamma f(V)}$.*

EXAMPLE 3. Let the finitely generated two-person NTU-game V be defined by

$$V(\{1\}) = V(\{2\}) = \{a \in \mathbf{R} \mid a \leq 3\}$$

and

$$V(\{1, 2\}) = \{(a, b) \in \mathbf{R}^2 \mid a \leq 4, b \leq 4, a + b \leq 5\}.$$

Then $V(\{i\}) = \bar{V}(\{i\}) = \tilde{V}(\{i\})$ for all $i \in \{1, 2\}$ and

$$\bar{V}(\{1, 2\}) = V(\{1, 2\}) \cup \{(a, b) \in \mathbf{R}^2 \mid a \leq 3, b \leq 3\}$$

$$\tilde{V}(\{1, 2\}) = \{(a, b) \in \mathbf{R}^2 \mid a \leq 4, b \leq 4, a + 2b \leq 9, 2a + b \leq 9\}$$

as represented in Fig. 1.

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